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Weak Convergence and Complete Convergence for $\widetilde{\varphi}$-mixing Sequences*

JIANG Yuan-ying, WU Qun-ying

(College of Science, Guilin University of Technology, Guilin 541004)

Abstract: In this article, we study some limit properties for sequences of $\widetilde{\varphi}$-mixing random variables. The classical weak law of large number and the Baum and Katz complete convergence theorem are established. They extend and improve the corresponding results from independent sequences of random variables to $\widetilde{\varphi}$-mixing sequences without imposing any unnecessarily extra conditions. Some well-known results are improved and extended.

Keywords: $\widetilde{\varphi}$-mixing random variable sequence; weak law of large numbers; complete convergence

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The random variables we deal with are all defined on $(\Omega, \mathcal{F}, P)$. Let $\{X_n;\ n \geq 1\}$ be a sequence of random variables. Denote $\mathcal{F}_S = \sigma(X_i,\ i \in S \subset N)$. Given σ-algebras $\mathcal{B}, \mathcal{R}$ in $\mathcal{F}$, let

$$\varphi(\mathcal{B}, \mathcal{R}) = \sup\left\{|P(B|A) - P(B)|;\ A \in \mathcal{B},\ P(A) > 0,\ B \in \mathcal{R}\right\},$$

and

$$\widetilde{\varphi}(n) = \sup\left\{\varphi(\mathcal{F}_S, \mathcal{F}_T);\ \text{finite subsets } S, T \subset N \text{ such that dist } (S,T) \geq n\right\}, \quad n \geq 0.$$

Obviously $0 \leq \widetilde{\varphi}(n+1) \leq \widetilde{\varphi}(n) \leq 1,\ n \geq 0$ and $\widetilde{\varphi}(0) = 1$.

Definition 1 A random variable sequence $\{X_n;\ n \geq 1\}$ is said to be a $\widetilde{\varphi}$-mixing random variable sequence if there exists an $k \in N$ such that $\widetilde{\varphi}(k) < 1$.

Without loss of generality, we assume that $\{X_n; n \geq 1\}$ is such that $\widetilde{\varphi}(1) < 1$. $\widetilde{\varphi}$-mixing is similar to φ-mixing, but they are quite different from each other. A number of researchers have studied mixing random variable sequences and a series of useful results have been established[1-4].

The main purpose of this paper is to study the weak convergence and complete convergence of $\widetilde{\varphi}$-mixing random variable sequences and try to obtain some new results. Our results extend and improve the corresponding results for independent random variable sequences to the case of $\widetilde{\varphi}$-mixing random variable sequences without adding any extra conditions.

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Lemma 1 Let $\{X_n;\, n \geq 1\}$ be a $\widetilde{\varphi}$-mixing random variable sequence with $EX_i = 0$ and $E|X_i|^q < \infty$ for every $i \geq 1$ and $q \geq 2$. Then there exists a positive constant c such that for all $n \geq 1$

$$E\Big(\max_{1\leq i\leq n} |S_i|^q\Big) \leq c\Big(\sum_{i=1}^{n} E|X_i|^q + \Big(\sum_{i=1}^{n} EX_i^{\,2}\Big)^{q/2}\Big),$$

where $S_i = \sum\limits_{j=1}^{i} X_j$.

Proof We can prove the Lemma by using the similar method as that for Theorem 2.1 of Sergey Utev and Magda Peligrad[7].

Lemma 2 Let $\{X_n;\, n \geq 1\}$ be a $\widetilde{\varphi}$-mixing random variable sequence. Then for any $x \geq 0$, there exists a positive constant c such that for all $n \geq 1$

$$\Big(1 - P\Big(\max_{1\leq k\leq n} |X_k| > x\Big)\Big)^2 \sum_{k=1}^{n} P(|X_k| > x) \leq cP\Big(\max_{1\leq k\leq n} |X_k| > x\Big).$$

Proof Let $A_k = (|X_k| > x)$ and

$$\alpha_n = 1 - P\Big(\bigcup_{k=1}^{n} A_k\Big) = 1 - P\Big(\max_{1\leq k\leq n} |X_k| > x\Big).$$

Without loss of generality, assume $\alpha_n > 0$. By the Cauchy-Schwarz inequality and Lemma 1, we have

$$\begin{aligned}
\sum_{k=1}^{n} P(A_k) &= \sum_{k=1}^{n} P\Big(A_k, \bigcup_{j=1}^{n} A_j\Big) = \sum_{k=1}^{n} E\Big(I_{A_k} I_{\bigcup\limits_{j=1}^{n} A_j}\Big) \\
&= E\Big(\sum_{k=1}^{n} (I_{A_k} - EI_{A_k})\Big) I_{\bigcup\limits_{j=1}^{n} A_j} + \sum_{k=1}^{n} P(A_k) P\Big(\bigcup_{j=1}^{n} A_j\Big) \\
&\leq \Big(\frac{c(1-\alpha_n)}{\alpha_n} \alpha_n \sum_{k=1}^{n} P(A_k)\Big)^{1/2} + (1-\alpha_n) \sum_{k=1}^{n} P(A_k) \\
&\leq \frac{1}{2}\Big(\frac{c(1-\alpha_n)}{\alpha_n} + \alpha_n \sum_{k=1}^{n} P(A_k)\Big) + (1-\alpha_n) \sum_{k=1}^{n} P(A_k).
\end{aligned}$$

Thus

$$\alpha_n^2 \sum_{k=1}^{n} P(A_k) \leq c(1-\alpha_n),$$

$$\Big(1 - P\Big(\max_{1\leq k\leq n} |X_k| > x\Big)\Big)^2 \sum_{k=1}^{n} P(A_k) \leq cP\Big(\max_{1\leq k\leq n} |X_k| > x\Big).$$

2 Weak convergence

In the following, let $a_n \ll b_n\, (a_n \gg b_n)$ denote that exists a constant $c > 0$ such that $a_n \leq cb_n\, (a_n \geq cb_n)$ for n sufficiently large, $\log x$ mean $\ln(\max(x, \mathrm{e}))$, and $S_n = \sum\limits_{j=1}^{n} X_j$.

Theorem 1 Let $\{X_n;\, n \geq 1\}$ be a $\widetilde{\varphi}$-mixing identically distributed random variable sequence (i.d.r.v.s.) satisfying

$$\lim_{n\to\infty} nP\big(|X_1| > n^p\big) = 0, \quad \text{for} \quad p > 1/2, \tag{1}$$

then

$$S_n/n^p - n^{1-p}EX_1I_{(|X_1|\leq n^p)} \xrightarrow{P} 0. \tag{2}$$

Remark 1 When $p = 1$ and $\{X_n;\, n \geq 1\}$ i.i.d, then Theorem 1 is the weak law of large numbers (WLLN) due to Feller[5]. So, Theorem 1 extends the sufficient part of Feller's WLLN for i.i.d.r.v.s. to a $\widetilde{\varphi}$-mixing setting.

Proof Let $X_j' = X_jI_{(|X_j|\leq n^p)}$ for $1 \leq j \leq n$ and $S_n' = \sum\limits_{j=1}^{n} X_j'$. Then, for each $n \geq 2$, $\{X_j';\, 1 \leq j \leq n\}$ are $\widetilde{\varphi}$-mixing i.d.r.v.s. and for every $\varepsilon > 0$

$$\begin{aligned} P\Big(\Big|\frac{S_n}{n^p} - \frac{S_n'}{n^p}\Big| > \varepsilon\Big) &= P\Big(\frac{S_n}{n^p} \neq \frac{S_n'}{n^p}\Big) = P\Big(\bigcup_{j=1}^{n}(X_j \neq X_j')\Big) \\ &\leq \sum_{j=1}^{n} P\big(|X_j| > n^p\big) = nP\big(|X_1| > n^p\big) \to 0, \end{aligned}$$

due to (1). So that (1) entails $S_n'/n^p - S_n/n^p \xrightarrow{P} 0$. Thus, to prove (2), it suffices to verify that

$$S_n'/n^p - n^{1-p}EX_1I_{(|X_1|\leq n^p)} \xrightarrow{P} 0. \tag{3}$$

By (1) and the Toeplity Lemma, we have

$$\frac{\sum\limits_{k=1}^{n} k^{2p-2}\cdot kP(|X_1| > k^p)}{\sum\limits_{j=1}^{n} j^{2p-2}} \to 0, \quad n \to \infty.$$

With this and $\sum\limits_{j=1}^{n} j^{2p-2} = O(n^{2p-1})$ for $p > 1/2$, we have

$$n^{-2p+1}\sum_{k=1}^{n} k^{2p-1}P\big(|X_1| > k^p\big) \to 0, \quad n \to \infty,$$

which, in conjunction with Lemma 1 for every $\varepsilon > 0$,

$$\begin{aligned} P\big(|S_n' - ES_n'| > \varepsilon n^p\big) &\ll n^{-2p}E(S_n' - ES_n')^2 = n^{-2p}E\Big(\sum_{j=1}^{n}(X_j' - EX_j')\Big)^2 \\ &\ll n^{-2p}\sum_{j=1}^{n} E(X_j' - EX_j')^2 \leq n^{-2p+1}EX_1'^2 = n^{-2p+1}EX_1^2I_{(|X_1|\leq n^p)} \\ &= n^{-2p+1}\sum_{k=1}^{n} EX_1^2I_{((k-1)^p<|X_1|\leq k^p)} \\ &= n^{-2p+1}\sum_{k=1}^{n} k^{2p}\big[P\big(|X_1| > (k-1)^p\big) - P\big(|X_1| > k^p\big)\big] \\ &\ll n^{-2p+1}\Big[\sum_{k=1}^{n} k^{2p-1}P\big(|X_1| > k^p\big) + 1\Big] \longrightarrow 0. \end{aligned}$$

Thus

$$(S_n' - ES_n')/n^p = S_n'/n^p - n^{1-p}EX_1 I_{(|X_1|\le n^p)} \xrightarrow{P} 0.$$

i.e. (3) holds.

3 Complete convergence

Definition 2 A function $l(x) > 0\,(x > 0)$ is said to be a slowly varying function if for any $c > 0$, $\lim\limits_{x\to\infty} l(cx)/l(x) = 1$.

Lemma 3[6] Let $l(x)$ be a slowly varying function, then

(i) $\lim\limits_{k\to\infty} \sup\limits_{2^k\le x<2^{k+1}} l(x)/l(2^k) = 1$.

(ii) For any $r > 0$, $\eta > 0$ and any natural number k, there exist constants c_1, $c_2 > 0$ such that

$$c_1 2^{kr} l(2^k\eta) \le \sum_{j=1}^{k} 2^{jr} l(2^j\eta) \le c_2 2^{kr} l(2^k\eta).$$

(iii) For any $r < 0$, $\eta > 0$ and any natural number k, there exist constants d_1, $d_2 > 0$ such that

$$d_1 2^{kr} l(2^k\eta) \le \sum_{j=k}^{\infty} 2^{jr} l(2^j\eta) \le d_2 2^{kr} l(2^k\eta).$$

Theorem 2 Let $\{X_n;\, n \ge 1\}$ be a $\widetilde{\varphi}$-mixing i.d.r.v.s., and $l(x)$ be a slowly varying function. Then for $0 < p < 2$, $\alpha p \ge 1$ and $EX_1 = 0$, the following statements are equivalent

$$E\big(|X_1|^p l(|X_1|^{1/\alpha})\big) < \infty, \tag{4}$$

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\Big(\max_{1\le j\le n} |S_j| > \varepsilon n^\alpha\Big) < \infty, \quad \forall\, \varepsilon > 0. \tag{5}$$

Remark 2 Take $l(x) \equiv 1$ and $\{X_n;\, n \ge 1\}$ i.i.d., Theorem 2 becomes the Baum and Katz[7] complete convergence theorem. So Theorem 2 extends and improves the Baum and Katz complete convergence theorem for i.i.d.r.v.s. to a $\widetilde{\varphi}$-mixing i.d.r.v.s.

Proof (4) $\Rightarrow$ (5). Let $Y_i = X_i I_{(|X_i|\le n^\alpha)}$, $i = 1, 2, \cdots, n$. Firstly, we prove that

$$n^{-\alpha} \max_{1\le j\le n} \Big|\sum_{i=1}^{j} EY_i\Big| \to 0, \quad n\to\infty. \tag{6}$$

(i) For $\alpha \le 1$, we have $p \ge 1/\alpha \ge 1$ and $EX_1 = 0$, $E|X_1|^p < \infty$ from (4). Thus

$$\begin{aligned} n^{-\alpha} \max_{1\le j\le n} \Big|\sum_{i=1}^{j} EY_i\Big| &\le n^{-\alpha} \sum_{i=1}^{n} |EY_i| = n^{1-\alpha} \big|EX_1 I_{(|X_1|>n^\alpha)}\big| \\ &\le n^{1-\alpha} E|X_1| \frac{|X_1|^{p-1}}{n^{\alpha(p-1)}} I_{(|X_1|>n^\alpha)} \ll n^{1-\alpha p} E|X_1|^p I_{(|X_1|>n^\alpha)} \to 0. \end{aligned}$$

(ii) For $\alpha > 1$, $p \ge 1$, we have by (4) that $E|X_1| < \infty$ because of the Jensen inequality, hence

$$n^{-\alpha} \max_{1\le j\le n} \Big|\sum_{i=1}^{j} EY_i\Big| \le n^{1-\alpha} E|X_1| I_{(|X_1|\le n^\alpha)} \ll n^{1-\alpha} \to 0.$$

(iii) For $\alpha > 1$, $p < 1$, since

$$n^{-\alpha}\max_{1\le j\le n}\Big|\sum_{i=1}^{j} EY_i\Big| \le n^{-\alpha}\sum_{i=1}^{n}|EY_i| = n^{1-\alpha}|EY_1|$$

$$= n^{1-\alpha}E|X_1|I_{(|X_1|\le n^\alpha)} = n^{1-\alpha}\sum_{i=1}^{n}E|X_1|I_{((i-1)^\alpha<|X_1|\le i^\alpha)}.$$

Noting $p < 1$, $\alpha p \ge 1$, we get

$$\sum_{i=1}^{\infty} i^{1-\alpha}E|X_1|I_{((i-1)^\alpha<|X_1|\le i^\alpha)} \le \sum_{i=1}^{\infty} i^{1-\alpha p}E|X_1|^p I_{((i-1)^\alpha<|X_1|\le i^\alpha)}$$

$$\le \sum_{i=1}^{\infty} E|X_1|^p I_{((i-1)^\alpha<|X_1|\le i^\alpha)} = E|X_1|^p < \infty.$$

By $i^{\alpha-1}\uparrow\infty$ and the Kronecker Lemma, we have

$$n^{1-\alpha}\sum_{i=1}^{n}E|X_1|I_{((i-1)^\alpha<|X_1|\le i^\alpha)} \to 0, \quad n\to\infty.$$

Hence (6) holds. So to prove (5), it suffices to prove that

$$\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)P\Big(\bigcup_{i=1}^{n}\big(|X_i|>n^\alpha\big)\Big) < \infty, \tag{7}$$

$$\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}(Y_i-EY_i)\Big| > \varepsilon n^\alpha\Big) < \infty, \quad \forall\,\varepsilon>0. \tag{8}$$

By Lemma 3 (i), (ii) and (4), it is easy to see that

$$\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)P\Big(\bigcup_{i=1}^{n}\big(|X_i|>n^\alpha\big)\Big)$$

$$\le \sum_{n=1}^{\infty} n^{\alpha p-1}l(n)P\big(|X_1|>n^\alpha\big) = \sum_{j=1}^{\infty}\sum_{2^j\le n<2^{j+1}} n^{\alpha p-1}l(n)P\big(|X_1|>n^\alpha\big)$$

$$\ll \sum_{j=1}^{\infty} 2^{j(\alpha p-1)}\cdot 2^j l(2^j)P\big(|X_1|>2^{\alpha j}\big) = \sum_{j=1}^{\infty}2^{\alpha pj}l(2^j)\sum_{k=j}^{\infty}P\big(2^{\alpha k}<|X_1|\le 2^{\alpha(k+1)}\big)$$

$$= \sum_{k=1}^{\infty}\sum_{j=1}^{k}2^{\alpha pj}l(2^j)P\big(2^{\alpha k}<|X_1|\le 2^{\alpha(k+1)}\big)$$

$$\ll \sum_{k=1}^{\infty}2^{\alpha pk}l(2^k)P\big(2^{\alpha k}<|X_1|\le 2^{\alpha(k+1)}\big) \ll E\big(|X_1|^p l(|X_1|^{1/\alpha})\big) < \infty,$$

i.e., (7) holds.

By the Markov inequality, Lemma 1, Lemma 3 (i), (iii) and (4), we obtain that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}(Y_i-EY_i)\Big|\ge \varepsilon n^{\alpha}\Big)\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} l(n)\sum_{i=1}^{n}E(Y_i-EY_i)^2 \le \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} l(n) EX_1^2 I_{(|X_1|\le n^{\alpha})}\\
&= \sum_{j=1}^{\infty}\sum_{2^{j-1}\le n<2^j} n^{\alpha p-1-2\alpha} l(n) EX_1^2 I_{(|X_1|\le n^{\alpha})} \ll \sum_{j=1}^{\infty} 2^{j\alpha(p-2)} l(2^j) EX_1^2 I_{(|X_1|\le 2^{\alpha j})}\\
&= \sum_{j=1}^{\infty} 2^{\alpha(p-2)j} l(2^j)\sum_{k=1}^{j} EX_1^2 I_{(2^{\alpha(k-1)}<|X_1|\le 2^{\alpha k})}\\
&= \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} 2^{\alpha(p-2)j} l(2^j) EX_1^2 I_{(2^{\alpha(k-1)}<|X_1|\le 2^{\alpha k})} \ll E\big(|X_1|^p l(|X_1|^{1/\alpha})\big)<\infty.
\end{aligned}$$

Now we prove that (5) $\Rightarrow$ (4). Obviously, (5) implies that

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\Big(\max_{1\le k\le n}|X_k|\ge \varepsilon n^{\alpha}\Big)<\infty, \quad \forall\, \varepsilon>0. \tag{9}$$

Noting $\alpha p-2>-1$, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} P\Big(\max_{1\le j\le 2^m}|X_j|\ge \varepsilon 2^{\alpha(m+1)}\Big)\\
&\ll \sum_{m=1}^{\infty}\sum_{2^m\le n<2^{m+1}} \frac{1}{n} P\Big(\max_{1\le j\le n}|X_j|> \varepsilon n^{\alpha}\Big) = \sum_{n=1}^{\infty}\frac{1}{n} P\Big(\max_{1\le j\le n}|X_j|\ge \varepsilon n^{\alpha}\Big)\\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\Big(\max_{1\le j\le n}|X_j|\ge \varepsilon n^{\alpha}\Big)<\infty.
\end{aligned}$$

Thus

$$\max_{2^{m-1}\le n<2^m} P\Big(\max_{1\le j\le n}|X_j|\ge \varepsilon 2^{2\alpha} n^{\alpha}\Big) \le P\Big(\max_{1\le j<2^m}|X_j|\ge \varepsilon 2^{\alpha(m+1)}\Big)\to 0.$$

Therefore, for n sufficiently large, we have

$$P\Big(\max_{1\le j\le n}|X_j|\ge 2\varepsilon n^{\alpha}\Big)<\frac{1}{2},$$

which, in conjunction with Lemma 2, means that

$$\sum_{k=1}^{n} P\big(|X_k|\ge \varepsilon 2^{2\alpha} n^{\alpha}\big) \le 4cP\Big(\max_{1\le j\le n}|X_j|\ge 2^{2\alpha}\varepsilon n^{\alpha}\Big).$$

Putting this into (9), we get

$$\sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P\big(|X_1|\ge 2^{2\alpha}\varepsilon n^{\alpha}\big)<\infty, \quad \forall\, \varepsilon>0.$$

Thus, by Lemma 3(i), (ii), we finally have

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P\big(|X_1| \geq 2^{2\alpha}\varepsilon n^{\alpha}\big) \\
&= \sum_{j=1}^{\infty} \sum_{2^j \leq n < 2^{j+1}} n^{\alpha p-1} l(n) P\big(|X_1| \geq 2^{2\alpha}\varepsilon n^{\alpha}\big) \\
&\gg \sum_{j=1}^{\infty} 2^{j\alpha p} l(2^j) P\big(|X_1| \geq \varepsilon 2^{\alpha} 2^{(j+1)\alpha} \hat{=} \varepsilon_0 2^{\alpha j}\big) \\
&= \sum_{j=1}^{\infty} 2^{\alpha p j} l(2^j) \sum_{k=j}^{\infty} P\big(\varepsilon_0 2^{\alpha k} \leq |X_1| < \varepsilon_0 2^{\alpha(k+1)}\big) \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{k} 2^{\alpha p j} l(2^j) P\big(\varepsilon_0 2^{\alpha k} \leq |X_1| < \varepsilon_0 2^{\alpha(k+1)}\big) \\
&\gg \sum_{k=1}^{\infty} 2^{\alpha p k} l(2^k) P\big(\varepsilon_0 2^{\alpha k} \leq |X_1| < \varepsilon_0 2^{\alpha(k+1)}\big) \gg E\big(|X_1|^p l(|X_1|^{1/\alpha})\big).
\end{aligned}
$$

It completes the proof of Theorem 2.

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$\widetilde{\varphi}$-混合序列的弱收敛性和完全收敛性

蒋远营， 吴群英

(桂林理工大学理学院，桂林 541004)

摘　要: 在本文中，作者研究了 $\widetilde{\varphi}$-混合序列的极限性质。得到了经典的弱收敛定理和完全收敛定理，将独立情形下的弱大数定律、Baum 和 Katz 完全收敛性定理推广到了 $\widetilde{\varphi}$-混合序列，而未额外添加任何条件。

关键词: $\widetilde{\varphi}$-混合随机变量序列；弱大数定律；完全收敛性